arXiv:0902.4881v1 [math.OC] 27 Feb 2009

Controllability and observability of an artificial advection-diffusion problem

Pierre Cornilleau, Sergio Guerrero
Ecole Centrale de Lyon, Université Paris 6

February 27, 2009

Abstract

In this paper we study the controllability of an artificial advection-diffusion system through the boundary. Suitable Carleman estimates give us the observability on the adjoint system in the one dimensional case. We also study some basic properties of our problem such as backward uniqueness and we get an intuitive result on the control cost for vanishing viscosity.

Introduction

Artificial advection-diffusion problem In the present paper we deal with some advection-diffusion problem with small viscosity truncated in one space direction. Our interest for the linear advection diffusion equation comes from the Navier-Stokes equation, but it arises also in other fields as, for example, meteorology. For a given viscosity $\varepsilon > 0$, the incompressible Navier-Stokes equation can be written as

$$\begin{cases} f_t + (f.\nabla)f - \varepsilon\Delta f + \nabla p = 0 \\ div(f) = 0 \end{cases}$$

where f is the velocity vector field, p the pressure, ∇ the gradient and Δ the usual Laplacian. Considering the flow around a body, we have that f is almost constant far away from the body and equal to a (see [10]). Linearizing the equation, we get the following equation for the vorticity

$$u_t + a.\nabla u - \varepsilon\Delta u = 0.$$

In the sequel, we assume for simplicity that a is the nth unit vector of the canonical basis of $\mathbb{R}^n$.

When one computes the solution of this problem, one can only solve numerically this problem on a bounded domain. A good way to approximate the solution on the whole space may be given by the use of artificial boundary conditions (see [6]). For any $T > 0$, we hence consider the following control problem on $\Omega = \mathbb{R}^{n-1} \times (-L, 0)$ (L some positive constant)

$$(S_v)\begin{cases} u_t + \partial_n u - \varepsilon\Delta u = 0 & \text{in } (0,T)\times\Omega, \\ \varepsilon(u_t + \partial_\nu u) = v & \text{on } (0,T)\times\Gamma_0, \\ \varepsilon(u_t + \partial_\nu u) + u = 0 & \text{on } (0,T)\times\Gamma_1, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases}$$

where $\Gamma_0 := \mathbb{R}^{n-1} \times \{0\}$ and $\Gamma_1 := \mathbb{R}^{n-1} \times \{-L\}$ forms a partition of the boundary $\partial\Omega$. Here we have denoted ∂_n the partial derivative with respect to x_n and ∂_ν the normal derivative.
We are interested in the so-called *null controllability* of this system

$$\text{for given } u_0, \text{find } v \text{ such that the solution of } (S_v) \text{ satisfies } u(T) \equiv 0.$$

Using classical duality arguments, we will be interested on the observability of the adjoint system through Γ_0.

Let us first introduce the adjoint system:

$$(S')\begin{cases} \varphi_t + \partial_n \varphi + \varepsilon \Delta \varphi = 0 & \text{in } (0,T)\times\Omega, \\ \varepsilon(\varphi_t - \partial_\nu \varphi) - \varphi = 0 & \text{on } (0,T)\times\Gamma_0, \\ \varphi_t - \partial_\nu \varphi = 0 & \text{on } (0,T)\times\Gamma_1. \\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{cases}$$

The observability inequality corresponding to the previous controllability property is:

$$\text{there exists } C>0 \text{ such that } \|\varphi(0,.)\|_X \le C\|\varphi\|_{L^2((0,T)\times\Gamma_0)} \quad \forall \varphi_T \in X, \tag{1}$$

where the space X will be defined below.

One can notice that, when the viscosity ε vanishes, our heat system (S_v) tends to a transport system with Dirichlet boundary condition on Γ_1. This phenomenon of degeneration of a parabolic problem to an hyperbolic one has been studied in several papers: see, for instance, [2] (one dimensional heat equation) and [5] (Burgers equation). Similar results of interests can also be found in [1].

Main results We define X as the closure of $\mathcal{D}(\bar{\Omega})$ for the norm $\|u\|_X := \left(\|u\|^2_{L^2(\Omega)} + \varepsilon\|u\|^2_{L^2(\partial\Omega)}\right)^{\frac{1}{2}}$.

We will denote by $C_{obs}(\varepsilon)$ the cost of the null-control, which is the smallest constant C which fulfills the observability estimate (1). Our main result of the paper is the following:

Theorem 1 *Assume $n=1$ and $T, L>0$.*

- *There exists $\overline{C}>0$ such that the solution of problem (S') satisfies (1). Consequently, for every $u_0 \in X$, there exists a control $v \in L^2((0,T)\times\Gamma_0)$ with*

$$\|v\|_{L^2((0,T)\times\Gamma_0)} \le \overline{C}\|u_0\|_X$$

  such that the solution of the problem (S_v) satisfies $u(T)\equiv 0$.

- *Furthermore, if $T/L-1$ is large enough, the cost of the null-control $C_{obs}(\varepsilon)$ tends to zero exponentially as $\varepsilon \to 0$.*

Remark 1 *One can in fact obtain an observability result for the adjoint system on Γ_1, that is*

$$\|\varphi(0,.)\|_X \le C\|\varphi\|_{L^2(\Gamma_0\times(0,T))},$$

for any φ solution of (S'). This provides some controllability result for the direct system on Γ_1: one can find a function v depending continuously on u_0 so that the solution of

$$\begin{cases} u_t + \partial_x u - \varepsilon\partial_{xx} u = 0 & \textit{in } (0,T)\times\Omega, \\ u_t + \partial_\nu u = 0 & \textit{on } (0,T)\times\Gamma_0, \\ \varepsilon(u_t + \partial_\nu u) + u = v & \textit{on } (0,T)\times\Gamma_1, \\ u(0,.) = u_0 & \textit{in } \Omega, \end{cases}$$

satisfies $u(T,.)\equiv 0$.

Remark 2 *The fact that the control cost vanishes tells intuitively that the state is almost null for $T/L-1$ big enough. This is to be connected with the fact that, for $\varepsilon=0$, the system is purely advective and then that, for $T>L$, its state is null.*

In some context of inverse problems (be able to know the origin of a polluted river for instance), it can be interesting to know if the observation of the solution of the direct problem on the boundary part Γ_1 or Γ_0 can allow us to recover the initial data. The corresponding result is presented now and will be proved at the end of the first section.

Proposition 2 *If $n=1$ and $T>0$, if the solution of (S_0) with initial data $u_0 \in X$ satisfies $u=0$ on $(0,T)\times\Gamma_1$, then $u_0\equiv 0$. However, there is no constant $C>0$ such that the following estimate holds true*

$$\|u_0\|_X \le C\|u\|_{L^2((0,T)\times\Gamma_1)} \quad \forall u_0 \in X. \tag{2}$$

Remark 3 *Using Remark 1, one can also obtain similar result for* Γ_0.

The rest of the article is organized as follows: in the first section, we show the well-posedness of the direct and the adjoint problems using some semi-groups approach. In the second section, we will adapt Carleman inequality to the case of our one-dimensional problem. The third section is intended to explain how to get observability in the one-dimensional case and the equivalence between observability and control.

Notation:
$A \lesssim B$ means that, for some universal constant $c > 0$, $A \leq cB$.
$A \sim B$ means that, for some universal constant $c > 1$, $c^{-1}B \leq A \leq cB$.

1 Well-posedness and basic properties of systems

1.1 Homogeneous problems

We will use some semi-group results to show existence and uniqueness of the homogeneous direct problem (that is (S_0)). This will enable to define solutions of the system (S_0) as a semigroup value. We define $H = (X, \|.\|_X), V = H^1(\Omega)$ endowed with the usual norm $\|.\|$ and we consider the bilinear form on V defined by

$$a(u_1, u_2) = \varepsilon \int_\Omega \nabla u_1.\nabla u_2 + \int_\Omega \partial_n u_1 u_2 + \int_{\Gamma_1} u_1 u_2. \tag{3}$$

Using Riesz representation theorem, one can define an operator $\mathcal{A}$ such that $a(u_1, u_2) =< -\mathcal{A}u_1, u_2 >_X$ and $\mathcal{D}(\mathcal{A}) = \{u \in X; -\varepsilon\Delta u + \partial_n u \in L^2(\Omega), \partial_\nu u \in L^2(\partial\Omega)\}$ equipped with the graph norm

$$\|u\|_{\mathcal{D}(\mathcal{A})} = \|u\|_X + \sup_{v \in X/\{0\}} \frac{|a(u,v)|}{\|v\|_X}.$$

(S_0) might now be written in the following abstract way

$$\begin{cases} \dot{u} = \mathcal{A}u & \text{in } X, \\ u(0,.) = u_0 & \text{in } X. \end{cases}$$

Proposition 3 $\mathcal{A}$ *generates an analytic semi-group* $(e^{t\mathcal{A}})_{t\geq 0}$ *on* X.

Proof Using Hille-Yoshida theorem, we will show that $\mathcal{A}$ is a maximal monotone operator.

- First, Green-Riemann formula gives

  $$< \mathcal{A}u, u >_X = -\varepsilon \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_{\partial\Omega} |u|^2,$$

  so the monotony is proved.

- Given $v \in X$, we have to solve $(I - \mathcal{A})u = v$, that is to find $u \in \mathcal{D}(\mathcal{A})$ such that

  $$\forall u' \in X, < (I - \mathcal{A})u, u' >_X = < v, u' >_X .$$

  The left-hand side term of this equation is a continuous bilinear form B on the space V while the right-hand side term is a continuous linear form L on V. Moreover, using (3), one can easily compute

  $$B(u,u) = \|u\|_X^2 + \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} |u|^2 \geq \min\{1, \varepsilon\}\|u\|^2,$$

  which show that B is coercive on V. Consequently, Lax-Milgram theorem shows that there exists $u \in V$ such that $B(u, u') = L(u')$, $\forall u' \in V$. Using test functions $u' \in \mathcal{D}(\Omega)$ additionnaly gives $u \in \mathcal{D}(\mathcal{A})$ and our proof ends.

□

In particular, for every initial data $u_0 \in X$, we have existence and uniqueness of a solution $u \in \mathcal{C}(\mathbb{R}^+, X)$ to (S_0). We will call these solutions *weak solutions* opposed to *strong solutions* i.e. such that $u_0 \in \mathcal{D}(\mathcal{A})$ and which fulfill $u \in \mathcal{C}(\mathbb{R}^+, \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}^+, X)$. One can notice that, using the density of $\mathcal{D}(\mathcal{A})$ in X, a weak solution can always be approximated by a strong solution.

Concerning the dual system, one can show that it may be written in the following abstract way

$$\begin{cases} \dot\varphi + \mathcal{A}^*\varphi = 0 & \text{in } X, \\ \varphi(T,.) = \varphi_T & \text{in } X. \end{cases}$$

In the same way as above, the adjoint operator $\mathcal{A}^*$ is shown to be maximal monotone with domain $\mathcal{D}(\mathcal{A}^*)$. Thus, for every initial data $\varphi_T \in X$, we have existence and uniqueness of a solution $\varphi \in \mathcal{C}(\mathbb{R}^+, X)$ to (S') given by means of the backward semigroup $(e^{(T-t)\mathcal{A}^*})_{t\geq 0}$. We will also speak of *weak solutions* or *strong solutions* in this situation.

1.2 Nonhomogeneous direct problems

We consider a slightly more general system

$$(S_{f,g_0,g_1}) \begin{cases} u_t + \partial_n u - \varepsilon\Delta u = f & \text{in } (0,T)\times\Omega, \\ \varepsilon(u_t + \partial_\nu u) = g_0 & \text{on } (0,T)\times\Gamma_0, \\ \varepsilon(u_t + \partial_\nu u) + u = g_1 & \text{on } (0,T)\times\Gamma_1, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases}$$

with $f \in L^2((0,T)\times\Omega)$, $g_0 \in L^2((0,T)\times\Gamma_0)$ and $g_1 \in L^2((0,T)\times\Gamma_1)$. We consider a function g on $\partial\Omega$ such that $g = g_i$ on Γ_i $(i = 0,1)$.

Definition 4 *We say that $u \in \mathcal{C}([0,T], X)$ is a solution of (S_{f,g_0,g_1}) if, for every function $\varphi \in \mathcal{C}([0,T], H^2(\Omega)) \cap \mathcal{C}^1([0,T], X)$, the following identity holds true*

$$\int_0^T\int_\Omega u(\varphi_t + \partial_n\varphi + \varepsilon\Delta\varphi) + \int_0^T\int_{\partial\Omega} u(\varepsilon(\varphi_t - \partial_\nu\varphi) - 1_{\Gamma_0}\varphi) + \int_0^T\int_\Omega f\varphi + \int_0^T\int_{\partial\Omega} g\varphi$$
$$= \int_\Omega u(T)\varphi(T) - \int_\Omega u_0\varphi(0) + \varepsilon\int_{\partial\Omega} u(T)\varphi(T) - \varepsilon\int_{\partial\Omega} u_0\varphi(0).$$

Proposition 5 *Let $T > 0$, $u_0 \in X$, $f \in L^2((0,T)\times\Omega)$, $g_0 \in L^2((0,T)\times\Gamma_0)$ and $g_1 \in L^2((0,T)\times\Gamma_1)$. Then (S_{f,g_0,g_1}) possesses a unique solution u. Moreover, one has the following estimate*

$$\|u\|_{L^\infty((0,T),X)} \leqslant C(T,\varepsilon)\left(\|u_0\|_X + \|f\|_{L^2(\Omega\times(0,T))} + \|g\|_{L^2(\partial\Omega\times(0,T))}\right),$$

with $C(T,\varepsilon)$ only depending on T and ε.

Proof Using Riesz representation theorem, we first define $F(t) \in X$ such that

$$< F(t), u >_X = \int_\Omega f(t)u + \int_{\partial\Omega} g(t)u, \quad \forall u \in X.$$

It is now easy to check that u is solution to (S_{f,g_0,g_1}) if and only if

$$u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}F(s)ds, \quad \forall t \in [0,T].$$

Finally, using classical estimates of the norm of $e^{t\mathcal{A}}$, this expression is well-defined and we obtain the required estimate. □

We now focus on some regularization effect for our general system and for some technical reasons we want to have some explicit dependence of the bounds on ε. The result is given below.

Lemma 6 *Let $\varepsilon \in (0,1)$, $f \in L^2((0,T)\times\Omega)$, $g_0 \in L^2((0,T)\times\Gamma_0)$ and $g_1 \in L^2((0,T)\times\Gamma_1)$.*

- *Let $u_0 \in X$. Then if u is the solution of (S_{f,g_0,g_1}), u belongs to $L^2((0,T),H^1(\Omega)) \cap \mathcal{C}([0,T],X)$ and*

$$\varepsilon^{1/2}\|u\|_{L^2((0,T),H^1(\Omega))} + \|u\|_{L^\infty((0,T),X)} \leq C(\|u_0\|_X + \|f\|_{L^2((0,T)\times\Omega)} + \|g\|_{L^2((0,T)\times\partial\Omega)}) \tag{4}$$

  for some $C > 0$ only depending on T.

- *Let now $u_0 \in H^1(\Omega)$. Then $u \in L^2((0,T),\mathcal{D}(\mathcal{A})) \cap \mathcal{C}([0,T],H^1(\Omega))$, $\partial_t u \in L^2((0,T),X)$ and*

$$\begin{aligned}&\varepsilon^{1/2}\|u\|_{L^\infty([0,T],H^1(\Omega))} + \varepsilon(\|\Delta u\|_{L^2((0,T)\times L^2(\Omega))} + \|\partial_\nu u\|_{L^2((0,T)\times\partial\Omega)})) + \|\partial_t u\|_{L^2([0,T],X)} \\ &\leq \frac{C}{\varepsilon^{1/2}}(\|u_0\|_{H^1(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)} + \|g\|_{L^2((0,T)\times\partial\Omega)}),\end{aligned} \tag{5}$$

  where $C > 0$ only depends on T.

Proof

- First, we multiply our main equation by u and we integrate on Ω. An application of Green-Riemann formula with use of the boundary conditions gives the identity

$$\frac{1}{2}\partial_t\|u\|_X^2 + \frac{1}{2}\int_{\partial\Omega}|u|^2 + \varepsilon\int_\Omega|\nabla u|^2 \ = \int_\Omega fu + \int_{\partial\Omega} gu,$$

  which immediately yields

$$\partial_t\|u\|_X^2 + \int_{\partial\Omega}|u|^2 + 2\varepsilon\int_\Omega|\nabla u|^2 \leq \|u\|_X^2 + \|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\partial\Omega)}^2.$$

  Finally, Gronwall's lemma yields (4).

- Now, we multiply the main equation by $\partial_t u$ and we integrate on Ω, which, using Green-Riemann formula and the boundary conditions provide the identity

$$\|\partial_t u\|_X^2 + \int_\Omega \partial_n u \partial_t u + \frac{\varepsilon}{2}\partial_t\left(\int_\Omega|\nabla u|^2\right) + \frac{1}{2}\partial_t\left(\int_\Omega|u|^2\right) = \int_\Omega f\partial_t u + \int_{\partial\Omega} g\partial_t u,$$

  Now

$$-\int_\Omega \partial_n u\partial_t u \leqslant \frac{1}{2}\int_\Omega|\nabla u|^2 + \frac{1}{2}\|\partial_t u\|_X^2$$

  and, using again Young's inequality, we have

$$\int_\Omega f\partial_t u + \int_{\partial\Omega} g\partial_t u \leqslant \frac{1}{4}\|\partial_t u\|_X^2 + C(\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\partial\Omega)}^2).$$

  We have thus proven the following inequality

$$\|\partial_t u\|_X^2 + 2\varepsilon\partial_t\left(\int_\Omega|\nabla u|^2\right) + 2\partial_t\int_{\Gamma_1}|u|^2 \leq 2\int_\Omega|\nabla u|^2 + C\left(\|g\|_{L^2(\partial\Omega)}^2 + \|f\|_{L^2(\Omega)}^2\right). \tag{6}$$

  Gronwall's lemma and (4) provide

$$\int_\Omega|\nabla u|^2 \leqslant \frac{C}{\varepsilon^2}\left(\int_0^T(\|f(t)\|_{L^2(\Omega)}^2 + \|g(t)\|_{L^2(\partial\Omega)}^2)dt + \|u_0\|_{H^1(\Omega)}^2\right), \tag{7}$$

  for $t \in (0,T)$. We now inject this into (6) to get the second part of the required result

$$\int_0^T\|\partial_t u(t)\|_X^2 dt \leqslant \frac{C}{\varepsilon}\left(\int_0^T(\|f(t)\|_{L^2(\Omega)}^2 + \|g(t)\|_{L^2(\partial\Omega)}^2)dt + \|u_0\|_{H^1(\Omega)}^2\right).$$

Writing the problem as

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon}(f - \partial_t u - \partial_n u) & \text{in } (0,T)\times\Omega, \\ \partial_\nu u = \frac{g_0}{\varepsilon} - \partial_t u & \text{on } (0,T)\times\Gamma_0, \\ \partial_\nu u = \frac{g_1}{\varepsilon} - \partial_t u - \frac{1}{\varepsilon} u & \text{on } (0,T)\times\Gamma_1, \end{cases}$$

we get that $u \in L^2((0,T),\mathcal{D}(\mathcal{A}))$ and the existence of some constant $C > 0$ such that

$$\begin{aligned} &\|\Delta u\|_{L^2((0,T)\times\Omega)} + \|\partial_\nu u\|_{L^2((0,T)\times\partial\Omega)} \\ &\le \frac{C}{\varepsilon}(\|f\|_{L^2((0,T)\times\Omega)} + \|g\|_{L^2((0,T)\times\partial\Omega)} + \|u\|_{L^2((0,T),H^1(\Omega))} + \|\partial_t u\|_{L^2((0,T),X)}) \end{aligned}$$

which gives the last part of the required result (5).

□

Remark 4 *Working on the non-homogeneous adjoint problem and using the backward semigroup $e^{(T-t)\mathcal{A}^*}$, one is able to show that the estimates of Lemma 6 hold true for the system*

$$(S'_{f,g_0,g_1})\begin{cases} \varphi_t + \partial_n\varphi + \varepsilon\Delta\varphi = f & \text{in } (0,T)\times\Omega, \\ \varepsilon(\varphi_t - \partial_\upsilon\varphi) - \varphi = g_0 & \text{on } (0,T)\times\Gamma_0, \\ \varepsilon(\varphi_t - \partial_\nu\varphi) = g_1 & \text{on } (0,T)\times\Gamma_1, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{cases}$$

1.3 Backward uniqueness

We will show here the backward uniqueness of systems (S_0) and (S'), thanks to the well-known result of Lions-Malgrange ([8]) and the regularization effect.

Lemma 7 *Let $u_0 \in X$ and $\delta \in (0,T)$. Then, the solution u of (S_0) satisfies $u \in \mathcal{C}([\delta,T],\mathcal{D}(\mathcal{A})$.*

Proof Our strategy is to show that $u \in L^2((t,T);H^2(\Omega))$ and $\partial_t u \in L^2((t,T);H^2(\Omega))$. This easily implies that $u \in \mathcal{C}([t,T];H^2(\Omega))$.

We first select some regular cut-off function θ_1 such that $\theta_1 = 1$ on $(\delta/2,T)$ and $\theta_1 = 0$ on $(0,\delta/4)$. If $u_1 = \theta_1 u$, we have that

$$\begin{cases} \partial_t u_1 + \partial_n u_1 - \varepsilon\Delta u_1 = \theta_1' u & \text{in } (0,T)\times\Omega, \\ \varepsilon(\partial_t u_1 + \partial_\nu u_1) = \varepsilon\theta_1' u & \text{on } (0,T)\times\Gamma_0, \\ \varepsilon(\partial_t u_1 + \partial_\nu u_1) + u_1 = \varepsilon\theta_1' u & \text{on } (0,T)\times\Gamma_1, \\ u_1(0,.) = 0 & \text{in } \Omega, \end{cases}$$

that is u_1 satisfies $(S_{\theta_1' u,\varepsilon\theta_1' u,\varepsilon\theta_1' u})$. An application of Lemma 6 gives that $u_1 \in L^2((0,T),\mathcal{D}(\mathcal{A}))$ and $\partial_t u_1 \in L^2((0,T),X)$. This implies in particular that $u \in L^2((\delta,T),\mathcal{D}(\mathcal{A}))$.

We now focus on $\partial_t u$ and we select another cut-off function θ_2 such that $\theta_2 = 1$ on (δ,T) and $\theta_2 = 0$ on $(0,\delta/2)$. We write the system satisfied by $\theta_2 u$ and we differentiate it with respect to time. One deduces that $u_2 = \partial_t(\theta_2 u)$ is the solution of

$$(S_{\theta_2' u' + \theta_2'' u,\varepsilon(\theta_2' u' + \theta_2'' u),\varepsilon(\theta_2' u' + \theta_2'' u)}).$$

Using that θ_2' is estimated by θ_1, Lemma 6 gives that $u_2 \in L^2((0,T),\mathcal{D}(\mathcal{A}))$ that is $\partial_t u \in L^2((\delta,T),\mathcal{D}(\mathcal{A}))$.

□

Proposition 8 *Assume that u is a weak solution of (S_0) such that $u(T) = 0$. Then $u_0 \equiv 0$.*

Proof If $0 < \delta < T$, then Lemma 7 shows that $u(\delta) \in \mathcal{D}(\mathcal{A})$. We will now apply Théorème 1.1 of [8] to u as a solution of (S_0) in the time interval (δ, T) to show that $u(\delta, .) = 0$. The bilinear form a defined in (3) can be split into two bilinear forms on V defined by

$$a(u_1, u_2) = a_0(t, u_1, u_2) + a_1(t, u_1, u_2)$$

with

$$a_0(t, u_1, u_2) = \varepsilon \int_\Omega \nabla u_1.\nabla u_2, \quad a_1(t, u_1, u_2) = \int_\Omega \partial_n u_1 u_2 + \int_{\Gamma_1} u_1 u_2.$$

The four first hypotheses in [8] (see (1.1)-(1.4) in that reference) are satisfied since $u \in L^2(\delta, T; V) \cap H^1(\delta, T; H)$ (from Lemma 6 above), $u(t) \in \mathcal{D}(\mathcal{A})$ for almost every $t \in (\delta, T)$, $u_t = \mathcal{A}u$ and $u(T, .) \equiv 0$. On the other hand,it is clear that a_0 and a_1 are continuous bilinear forms on V and that they do not depend on time t. It is also straightforward to see that

$$\forall u \in V,\ a_0(t, u, u) + \|u\|_X^2 \geqslant \min\{1, \varepsilon\}\|u\|^2$$

and, for some constant $C > 0$,

$$\forall u, v \in V,\ |a_1(t, u_1, u_2)| \leqslant C\|u_1\|\|u_2\|_X.$$

This means that Hypothesis I in that reference is fulfilled. We have shown that $u(\delta, .) = 0$, which finishes the proof since

$$u_0 = \lim_{\delta \to 0} u(\delta) = 0.$$

□

Remark 5 *The result also holds for the adjoint system: if φ is a weak solution of (S') such that $\varphi(0) \equiv 0$ then $\varphi_T = 0$. The proof is very similar to the one above and left to the reader.*

1.4 Proof of Proposition 2

The fact that $u = 0$ on $(0, T) \times \Gamma_1$ implies $u_0 \equiv 0$ is a straightforward consequence of the backward uniqueness of system (S_0). Indeed, similarly as (1), one can prove the observability inequality

$$\|u(T, .)\|_X \lesssim \|u\|_{L^2((0,T)\times\Gamma_1)},$$

(see Remark 6 below), which combined with Proposition 8 gives $u_0 \equiv 0$.
To show that (2) is false, we first need to prove some well-posedness of (S_0) with u_0 in a less regular space. For $s \in]0, 1/2[$, we define X^s as the closure of $\mathcal{D}(\bar{\Omega})$ for the norm

$$\|u\|_{X^s} := \left(\|u\|^2_{H^s(\Omega)} + \varepsilon \|u\|^2_{H^s(\partial\Omega)} \right)^{\frac{1}{2}},$$

where $H^s(\Omega)$ (resp. $H^s(\partial\Omega)$) stand for the usual Sobolev space on Ω (resp. $\partial\Omega$). One easily shows that X^s is a Hilbert space. We now denote X^{-s} the set of functions u_2 such that the linear form

$$u_1 \in \mathcal{D}(\bar{\Omega}) \longmapsto < u_1, u_2 >_X$$

can be extended in a continuous way on X^s. If $u_1 \in X^{-s}$, $u_2 \in X^s$ we denote this extension by $< u_1, u_2 >_{-s,s}$.

If $u_0 \in X^{-s}$, we say that u is a **solution by transposition** of (S_0) if, for every $f \in L^2((0, T) \times \Omega)$

$$\int_0^T \int_\Omega uf + < u_0, \varphi(0, .) >_{-s,s} = 0,$$

where φ is the weak solution (see Remark 4) of

$$(S'_{f,0,0}) \begin{cases} \varphi_t + \partial_n \varphi + \varepsilon \Delta \varphi = f & \text{in } (0, T) \times \Omega, \\ \varepsilon(\varphi_t - \partial_\nu \varphi) - \varphi = 0 & \text{on } (0, T) \times \Gamma_0, \\ \varphi_t - \partial_\nu \varphi = 0 & \text{on } (0, T) \times \Gamma_1, \\ \varphi(T, .) = 0 & \text{in } \Omega. \end{cases}$$

Using the Riesz representation theorem and continuity (see Remark 4) of

$$f \in L^2((0,T)\times\Omega) \longmapsto \varphi(0,.) \in H^1(\Omega),$$

it is obvious that, for any $u_0 \in X^{-s}$, there exists a solution by transposition $u \in L^2((0,T)\times\Omega)$ such that

$$\|u\|_{L^2((0,T)\times\Omega)} \lesssim \|u_0\|_{X^{-s}}.$$

Moreover, if $u_0 \in X = X^0$, Lemma 6 shows that the solution u satisfies

$$\|u\|_{L^2((0,T),H^1(\Omega))} \lesssim \|u_0\|_{X^0}.$$

If one uses classical interpolation results (see [7]), we can deduce that for every $\theta \in]1/2,1]$, for any $u_0 \in X^{-s\theta} = [X, X^{-s}]_{1-\theta}$, there exists a solution $u \in L^2((0,T),H^\theta(\Omega))$ such that

$$\|u\|_{L^2((0,T),H^\theta(\Omega))} \lesssim \|u_0\|_{X^{-s(1-\theta)}}.$$

Using classical trace result, we have that

$$\|u\|_{L^2((0,T)\times\partial\Omega)} \lesssim \|u_0\|_{X^{-s(1-\theta)}}. \tag{8}$$

On the other hand, estimate (2) imply in particular that

$$\|u_0\|_X \lesssim \|u\|_{L^2((0,T)\times\Gamma_1)} \quad \forall u_0 \in \mathcal{D}(\overline{\Omega}). \tag{9}$$

Finally, (8) and (9) yields the contradictory inclusion $X^{-s(1-\theta)} \hookrightarrow X$. $\square$

2 Carleman inequality in dimension 1

In this paragraph, we will establish a Carleman-type inequality keeping track of the explicit dependence of all the constants with respect to T and ε. As in [4], we introduce the following weight functions:

$$\eta(x) := 2L + x, \quad \alpha(t,x) := \frac{C - e^{\eta(x)}}{\varepsilon^2 t(T-t)}, \quad \phi(t,x) := \frac{e^{\eta(x)}}{\varepsilon^2 t(T-t)},$$

where $C > e^{2L}$.
The rest of this paragraph will be dedicated to the proof of the following inequality:

Theorem 9 *There exists $C > 0$ and $s_0 > 0$ such that for every $\varepsilon \in (0,1)$ and every $s \geqslant s_0(\varepsilon T + \varepsilon^2 T^2)$ the following inequality is satisfied for every $\varphi_T \in X$:*

$$\begin{aligned} &s^3 \int_{(0,T)\times(-L,0)} \phi^3 e^{-2s\alpha}|\varphi|^2 + s\int_{(0,T)\times(-L,0)} \phi e^{-2s\alpha}|\varphi_x|^2 + s^3 \int_{(0,T)\times\{0,-L\}} \phi^3 e^{-2s\alpha}|\varphi|^2 \\ &\leqslant Cs^9 \int_{(0,T)\times\{0\}} e^{-4s\alpha+2s\alpha(.,-L)}\phi^9|\varphi|^2. \end{aligned} \tag{10}$$

Here, φ stands for the solution of (S') associated to φ_T.

Remark 6 *One can in fact obtain the following Carleman estimate with control term in Γ_1*

$$\begin{aligned} &s^3 \int_{(0,T)\times(-L,0)} \phi^3 e^{-2s\alpha}|\varphi|^2 + s^2\int_{(0,T)\times(-L,0)} \phi e^{-2s\alpha}|\varphi_x|^2 + s^3 \int_{(0,T)\times\{0,-L\}} \phi^3 e^{-2s\alpha}|\varphi|^2 \\ &\leqslant Cs^9 \int_{(0,T)\times\{-L\}} e^{-4s\alpha+2s\alpha(.,0)}\phi^9|\varphi|^2. \end{aligned}$$

simply by choosing the weight function $\eta(x)$ equal to $x \mapsto -x + L$. The proof is very similar and the sequel will show how to deal with this case too.

In order to perform a Carleman inequality for system (S'), we first do a scaling in time. Introducing $\tilde{T} := \varepsilon T$, $\tilde{\varphi}(t,x) := \varphi(t/\varepsilon, x)$ and the weights $\tilde{\alpha}(t,x) := \alpha(t/\varepsilon, x)$, $\tilde{\phi}(t,x) := \phi(t/\varepsilon, x)$; we have the following system:

$$\begin{cases} \tilde{\varphi}_t + \varepsilon^{-1}\tilde{\varphi}_x + \tilde{\varphi}_{xx} = 0 & \text{in } q, \\ \varepsilon^2(\tilde{\varphi}_t - \varepsilon^{-1}\partial_\nu \tilde{\varphi}) - \tilde{\varphi} = 0 & \text{on } \sigma_0, \\ \tilde{\varphi}_t - \varepsilon^{-1}\partial_\nu \tilde{\varphi} = 0 & \text{on } \sigma_1, \\ \tilde{\varphi}_{|t=T} = \tilde{\varphi}_T & \text{in } (-L,0). \end{cases} \tag{11}$$

if $q := (0,\tilde{T}) \times (-L,0)$, $\sigma := \partial\Omega \times (-L,0)$, $\sigma_0 := \Gamma_0 \times (-L,0)$ and $\sigma_1 := \Gamma_1 \times (-L,0)$. We will now explain how to get the following result.

Proposition 10 *There exists $C > 0$, such that for every $\varepsilon \in (0,1)$ and every $s \geqslant C(\tilde{T} + \tilde{T}^2 + \varepsilon^{-1}\tilde{T}^2 + \varepsilon^{1/3}\tilde{T}^{2/3})$, the following inequality is satisfied for every solution of (11) associated to $\tilde{\varphi}_T \in X$:*

$$\begin{aligned} & s^3 \int_q \tilde{\phi}^3 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|^2 + s \int_q \tilde{\phi} e^{-2s\tilde{\alpha}} |\tilde{\varphi}_x|^2 + s^3 \int_\sigma \tilde{\phi}^3 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|^2 \\ & \leqslant C s^9 \int_{\sigma_0} e^{-4s\tilde{\alpha} + 2s\tilde{\alpha}(.,-L)} \tilde{\phi}^9 |\tilde{\varphi}|^2. \end{aligned} \tag{12}$$

Observe that Proposition 10 directly implies Theorem 9.

All along the proof we will need several properties of the weight functions:

Lemma 11

- $|\tilde{\alpha}_t| \lesssim \tilde{T}\tilde{\phi}^2$, $|\tilde{\alpha}_{xt}| \lesssim \tilde{T}\tilde{\phi}^2$, $|\tilde{\alpha}_{tt}| \lesssim \tilde{T}^2\tilde{\phi}^3$,
- $\tilde{\alpha}_x = -\tilde{\phi}$, $\tilde{\alpha}_{xx} = -\tilde{\phi}$.

There are essentially two steps in this proof: the first one consists of doing a Carleman estimate very similar to that of [4]; in the second one, we will study the boundary terms appearing in the right hand side due to the boundary conditions. We will perform the proof of this theorem for smooth solutions, so that the general proof follows from a density argument.

We will first estimate the left hand side terms of (12) like in the classical Carleman estimate (see [4]). We obtain:

Proposition 12 *There exists $C > 0$, such that for every $\varepsilon \in (0,1)$ and every $s \geqslant C(\tilde{T} + \tilde{T}^2 + \varepsilon^{-1}\tilde{T}^2 + \varepsilon^{1/3}\tilde{T}^{2/3})$, the following inequality is satisfied for every solution of (11) associated to $\tilde{\varphi}_T \in X$:*

$$\begin{aligned} & s^3 \int_q \tilde{\phi}^3 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|^2 + s^2 \int_q \tilde{\phi} e^{-2s\tilde{\alpha}} |\tilde{\varphi}_x|^2 + s^{-1} \int_q \tilde{\phi}^{-1} e^{-2s\tilde{\alpha}} (|\tilde{\varphi}_{xx}|^2 + |\tilde{\varphi}_t|^2) \\ & + s^3 \int_{\sigma_1} \tilde{\phi}^3 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|^2 \lesssim s^5 \int_{\sigma_0} \tilde{\phi}^5 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|^2 + (s+\varepsilon) \int_{\sigma_0} \tilde{\phi} e^{-2s\tilde{\alpha}} |\tilde{\varphi}_t|^2. \end{aligned} \tag{13}$$

Proof

Let us introduce $\psi := \tilde{\varphi} e^{-s\tilde{\alpha}}$. We state the equations satisfied by ψ. In $(0,\tilde{T}) \times (-L,0)$, we have the identity

$$P_1\psi + P_2\psi = P_3\psi,$$

where

$$P_1\psi = \psi_t + 2s\tilde{\alpha}_x\psi_x + \varepsilon^{-1}\psi_x, \tag{14}$$

$$P_2\psi = \psi_{xx} + s^2\tilde{\alpha}_x^2\psi + s\tilde{\alpha}_t\psi + \varepsilon^{-1}s\tilde{\alpha}_x\psi, \tag{15}$$

and

$$P_3\psi = s\tilde{\alpha}_{xx}\psi.$$

On the other hand, the boundary conditions are:

$$\varepsilon^2(\psi_t + s\tilde{\alpha}_t\psi - \varepsilon^{-1}(\psi_x + s\tilde{\alpha}_x\psi)) - \psi = 0 \quad \text{on } x = 0, \tag{16}$$

$$\varepsilon(\psi_t + s\tilde{\alpha}_t\psi) + \psi_x + s\tilde{\alpha}_x\psi = 0 \text{ on } x = -L. \tag{17}$$

We take the L^2 norm in both sides of the identity in q:

$$\|P_1\psi\|^2_{L^2(q)} + \|P_2\psi\|^2_{L^2(q)} + 2(P_1\psi, P_2\psi)_{L^2(q)} = \|P_3\psi\|^2_{L^2(q)}. \tag{18}$$

Using Proposition 11, we directly obtain

$$\| P_3\psi\|^2_{L^2(q)} \lesssim s^2 \int_q \tilde{\phi}^2|\psi|^2.$$

We focus on the expression of the double product $(P_1\psi, P_2\psi)_{L^2(q)}$. This product contains 12 terms which will be denoted by $T_{ij}(\psi)$ for $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 4$. We study them successively.

- An integration by parts in space and then in time shows that, since $\psi(0,.) = \psi(\tilde{T},.) = 0$, we have

$$\begin{aligned} T_{11}(\psi) = \int_q \psi_t\psi_{xx} = -\frac{1}{2}\int_q (|\psi_x|^2)_t + \int_{\sigma_0} \psi_x\psi_t - \int_{\sigma_1} \psi_x\psi_t \\ = \int_{\sigma_0} \psi_x\psi_t - \int_{\sigma_1} \psi_x\psi_t. \end{aligned}$$

  Now, we use the boundary condition (16). We have

$$\int_{\sigma_0} \psi_x\psi_t \gtrsim \varepsilon \int_{\sigma_0} |\psi_t|^2 - \varepsilon\tilde{T}^2 s \int_{\sigma_0} \tilde{\phi}^3|\psi|^2 - \tilde{T}s \int_{\sigma_0} \tilde{\phi}^2|\psi|^2.$$

  Thanks to the fact that $\psi(0,.) = \psi(\tilde{T},.) = 0$, the same computations can be done on $x = -L$ using (17) so we obtain the following for this term:

$$T_{11}(\psi) \gtrsim -\varepsilon\tilde{T}^2 s \int_\sigma \tilde{\phi}^3|\psi|^2 - \tilde{T}s \int_\sigma \tilde{\phi}^2|\psi|^2.$$

- Integrating by parts in time, we find

$$\begin{aligned} T_{12}(\psi) &= \frac{s^2}{2}\int_q \tilde{\alpha}_x^2(|\psi|^2)_t = -s^2\int_q \tilde{\alpha}_x\tilde{\alpha}_{xt}|\psi|^2, \\ T_{13}(\psi) &= \frac{s}{2}\int_q \tilde{\alpha}_t(|\psi|^2)_t = -\frac{s}{2}\int_q \tilde{\alpha}_{tt}|\psi|^2, \\ T_{14}(\psi) &= \varepsilon^{-1}\frac{s}{2}\int_q \tilde{\alpha}_x(|\psi|^2)_t = -\varepsilon^{-1}\frac{s}{2}\int_q \tilde{\alpha}_{tx}|\psi|^2, \end{aligned}$$

  and using Proposition 11, we get

$$\begin{aligned} T_{12}(\psi) &\gtrsim -\tilde{T}s^2\int_q \tilde{\phi}^3|\psi|^2, \\ T_{13}(\psi) &\gtrsim -\tilde{T}^2 s\int_q \tilde{\phi}^3|\psi|^2, \\ T_{14}(\psi) &\gtrsim -\varepsilon^{-1}\tilde{T}s\int_q \tilde{\phi}^2|\psi|^2. \end{aligned}$$

- Now, integrating by parts in space, we have

$$T_{21}(\psi) = s\int_q \tilde{\alpha}_x(|\psi_x|^2)_x = -s\int_q \tilde{\alpha}_{xx}|\psi_x|^2 + s\int_{\sigma_0} (\tilde{\alpha}_x|\psi_x|^2 - s\int_{\sigma_1} \tilde{\alpha}_x|\psi_x|^2.$$

  Thanks to Proposition 11 (the choice of η is important here), the last term is positive. Using the boundary condition (16) and Proposition 11, we finally get

$$\begin{aligned} T_{21}(\psi) \gtrsim s^2\int_q \tilde{\phi}|\psi_x|^2 - s^3\int_{\sigma_0} \tilde{\phi}^3|\psi|^2 - \varepsilon^2 s^3\tilde{T}^2\int_{\sigma_0} \tilde{\phi}^5|\psi|^2 \\ -\varepsilon^2 s\int_{\sigma_0} \tilde{\phi}|\psi_t|^2. \end{aligned}$$

- An integration by parts in space provides

$$T_{22}(\psi) = s^3 \int_q \tilde{\alpha}_x^3 (|\psi|^2)_x = -3s^3 \int_q \tilde{\alpha}_x^2 \tilde{\alpha}_{xx} |\psi|^2 + s^3 \int_{\sigma_0} \tilde{\alpha}_x^3 |\psi|^2$$
$$-s^3 \int_{\sigma_1} \tilde{\alpha}_x^3 |\psi|^2.$$

  This readily yields

$$T_{22}(\psi) \gtrsim s^3 \int_q \tilde{\phi}^3 |\psi|^2 + s^3 \int_{\sigma_1} \tilde{\phi}^3 |\psi|^2 - s^3 \int_{\sigma_0} \tilde{\phi}^3 |\psi|^2.$$

- Again an integration by parts in space gives

$$T_{23}(\psi) = s^2 \int_q \tilde{\alpha}_t \tilde{\alpha}_x (|\psi|^2)_x = -s^2 \int_q (\tilde{\alpha}_t \tilde{\alpha}_x)_x |\psi|^2 + s^2 \int_{\sigma_0} \tilde{\alpha}_t \tilde{\alpha}_x |\psi|^2$$
$$-s^2 \int_{\sigma_1} \tilde{\alpha}_t \tilde{\alpha}_x |\psi|^2.$$

  Using Proposition 11, we obtain

$$T_{23}(\psi) \gtrsim -s^2 \tilde{T} \int_q \tilde{\phi}^3 |\psi|^2 - s^2 \tilde{T} \int_{\sigma_0} \tilde{\phi}^3 |\psi|^2 - s^2 \tilde{T} \int_{\sigma_1} \tilde{\phi}^3 |\psi|^2.$$

- The last integral concerning the second term in the expression of $P_1\psi$ is

$$T_{24}(\psi) = -\varepsilon^{-1} s^2 \int_q \tilde{\alpha}_x^2 (|\psi|^2)_x.$$

  After an integration by parts in space, we get

$$T_{24}(\psi) \geq \varepsilon^{-1} s^2 \int_q \tilde{\phi}^2 |\psi|^2 - \varepsilon^{-1} s^2 \int_{\sigma_0} \tilde{\phi}^2 |\psi|^2.$$

- We consider now the third term in the expression of $P_1\psi$. First, we have

$$T_{31}(\psi) = -\frac{\varepsilon^{-1}}{2} \int_q (|\psi_x|^2)_x \geq -\frac{\varepsilon^{-1}}{2} \int_{\sigma_0} |\psi_x|^2.$$

  Using the boundary condition (16), we get

$$T_{31}(\psi) \gtrsim -\varepsilon \int_{\sigma_0} |\psi_t|^2 - \varepsilon s^2 \tilde{T}^2 \int_{\sigma_0} \tilde{\phi}^4 |\psi|^2 - \varepsilon^{-1} s^2 \int_{\sigma_0} \tilde{\phi}^2 |\psi|^2$$
$$-\varepsilon^{-3} \int_{\sigma_0} |\psi|^2.$$

- Now, we integrate by parts with respect to x and we have

$$T_{32}(\psi) = -\frac{\varepsilon^{-1}}{2} s^2 \int_q \tilde{\alpha}_x^2 (|\psi|^2)_x \geq \varepsilon^{-1} s^2 \int_q \tilde{\alpha}_x \tilde{\alpha}_{xx} |\psi|^2 - \frac{\varepsilon^{-1}}{2} s^2 \int_{\sigma_0} \tilde{\alpha}_x^2 |\psi|^2$$
$$\gtrsim -\varepsilon^{-1} s^2 \int_{\sigma_0} \tilde{\phi}^2 |\psi|^2.$$

- Then, using another integration by parts in x we obtain

$$T_{33}(\psi) = -\frac{\varepsilon^{-1}}{2} s \int_q \tilde{\alpha}_t (|\psi|^2)_x \gtrsim -\varepsilon^{-1} s \tilde{T} \int_q \tilde{\phi}^2 |\psi|^2 - \varepsilon^{-1} \tilde{T} s \int_{\sigma_0} \tilde{\phi}^2 |\psi|^2 - \varepsilon^{-1} \tilde{T} s \int_{\sigma_1} \tilde{\phi}^2 |\psi|^2.$$

- Finally, arguing as before, we find

$$T_{34}(\psi) = \frac{\varepsilon^{-2}}{2} s \int_q \tilde{\alpha}_x (|\psi|^2)_x \geq \frac{\varepsilon^{-2}}{2} s^2 \int_q \tilde{\phi}|\psi|^2 - C\varepsilon^{-2} s \int_{\sigma_0} \tilde{\phi}|\psi|^2$$
$$\gtrsim -\varepsilon^{-2} s \int_{\sigma_0} \tilde{\phi}|\psi|^2.$$

Putting together all the terms and combining the resulting inequality with (18), we obtain

$$\|P_1\psi\|^2_{L^2(q)} + \|P_2\psi\|^2_{L^2(q)} + I_1(\psi) + I_2(\psi_x) \lesssim J_1(\psi) + J_2(\psi_x) + J_3(\psi_t) + L(\psi) \tag{19}$$

where the main terms are

$$I_1(\psi) = s^3 \int_q \tilde{\phi}^3|\psi|^2 + s^3 \int_{\sigma_1} \tilde{\phi}^3|\psi|^2, \quad I_2(\psi_x) = s^2 \int_q \tilde{\phi}|\psi_x|^2,$$

the right hand side terms are

$$J_1(\psi) = (s^2 + s\tilde{T}^2 + \varepsilon^{-1}\tilde{T}s) \int_q \tilde{\phi}^2|\psi|^2 + (s^2\tilde{T} + s\tilde{T}^2) \int_q \tilde{\phi}^3|\psi|^2$$
$$+ s\tilde{T}^2 \int_{\sigma_1} \tilde{\phi}^2|\psi|^2 + (s^2\tilde{T} + \varepsilon s\tilde{T}^2) \int_{\sigma_1} \tilde{\phi}^3|\psi|^2 + (s\tilde{T} + \varepsilon^{-1}\tilde{T}s) \int_{\sigma_1} \tilde{\phi}^2|\psi|^2,$$

and

$$J_2(\psi_x) = \int_q |\psi_x|^2, \quad J_3(\psi_t) = (s + \varepsilon + \varepsilon^2 s) \int_{\sigma_0} \tilde{\phi}|\psi_t|^2,$$

and the control terms are

$$L(\psi) = (s\tilde{T} + \varepsilon^{-1}(\tilde{T}s + s^2)) \int_{\sigma_0} \tilde{\phi}^2|\psi|^2 + (s\tilde{T}^2 + s^3 + s^2\tilde{T} + \varepsilon\tilde{T}^2 s) \int_{\sigma_0} \tilde{\phi}^3|\psi|^2$$
$$+ \varepsilon s^2\tilde{T}^2 \int_{\sigma_0} \tilde{\phi}^4|\psi|^2 + (s^3\tilde{T}^2 + \varepsilon^2 s^3\tilde{T}^2) \int_{\sigma_0} \tilde{\phi}^5|\psi|^2 + \varepsilon^{-3} \int_{\sigma_0} (1 + \varepsilon s\tilde{\phi})|\psi|^2.$$

Let us now see that we can absorb some right hand side terms with the help of the parameter s.

• First, we see that the distributed terms in $J_1(\psi)$ can be absorbed by the first term in the definition of $I_1(\psi)$ for a choice of $s \gtrsim \tilde{T} + (1 + \varepsilon^{-1/2})\tilde{T}^{3/2}$.

• Then, we use the second term of $I_1(\psi)$ in order to absorb the integrals in the second line of the definition of $J_1(\psi)$. We find that this can be done as long as $s \gtrsim \tilde{T} + \tilde{T}^2 + (\varepsilon^{-1/2} + 1)\tilde{T}^{3/2}$.

• Next, $J_2(\psi_x)$ can be absorbed by $I_2(\psi_x)$ by just taking $s \gtrsim \tilde{T}^2$.

Then, we observe that all the control terms can be bounded in the following way:

$$|L(\psi)| \lesssim s^5 \int_{\sigma_0} \tilde{\phi}^5|\psi|^2$$

as long as

$$s \gtrsim \tilde{T}(1 + \varepsilon + \tilde{T}^{1/3}\varepsilon^{1/3} + \tilde{T}^{3/4}\varepsilon^{-1/4}) + \tilde{T}^2(1 + \varepsilon^{-3/5} + \varepsilon^{-1/2} + \varepsilon^{-1/3}).$$

Since $\varepsilon < 1$, it suffices to take $s \gtrsim \tilde{T}(1 + \varepsilon^{-1}\tilde{T})$.

Next, we use the expression of $P_1\psi$ and $P_2\psi$ (see (14)-(15)) in order to obtain some estimates for the terms ψ_t and ψ_{xx} respectively:

$$s^{-1} \int_q \tilde{\phi}^{-1}|\psi_t|^2 \lesssim s^2 \int_q \tilde{\phi}|\psi_x|^2 + s^{-1}\varepsilon^{-2} \int_q \tilde{\phi}^{-1}|\psi_x|^2 + s^{-1} \int_q \tilde{\phi}^{-1}|P_1\psi|^2$$
$$\lesssim s^2 \int_q \tilde{\phi}|\psi_x|^2 + \int_q |P_1\psi|^2$$

for any $s \gtrsim (1+\varepsilon^{-1})\tilde{T}^2$ and

$$s^{-1}\int_q \tilde{\phi}^{-1}|\psi_{xx}|^2 \lesssim s^3\int_0^{\tilde{T}}\int_{-L}^0 \tilde{\phi}^3|\psi|^2 + s\tilde{T}^2\int_q \tilde{\phi}^3|\psi_x|^2 + \varepsilon^{-2}s^2\int_q \tilde{\phi}|\psi|^2$$
$$+s^{-1}\int_q \tilde{\phi}^{-1}|P_2\psi|^2 \lesssim s^3\int_q \tilde{\phi}^3|\psi|^2 + s^2\int_q \tilde{\phi}|\psi_x|^2 + \int_q |P_2\psi|^2,$$

for $s \gtrsim \tilde{T} + \varepsilon^{-1}\tilde{T}^2$.
Combining all this with (19), we obtain

$$\begin{aligned} &s^2\int_q \tilde{\phi}(s^2\tilde{\phi}^2|\psi| + |\psi_x|^2) + s^{-1}\int_q \tilde{\phi}^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + s^3\int_{\sigma_1}\tilde{\phi}^3|\psi|^2 \\ &\lesssim s^5\int_{\sigma_0}\tilde{\phi}^5|\psi|^2 + (s+\varepsilon+\varepsilon^2 s)\int_{\sigma_0}\tilde{\phi}|\psi_t|^2, \end{aligned} \tag{20}$$

for $s \gtrsim \tilde{T}(1+\varepsilon^{-1}\tilde{T})$.

Finally, we come back to our variable $\tilde{\varphi}$. We first remark that $\psi_x = e^{-s\tilde{\alpha}}(\tilde{\varphi}_x + s\tilde{\phi}\tilde{\varphi})$ and so

$$s^2\int_q \tilde{\phi}e^{-2s\tilde{\alpha}}|\tilde{\varphi}_x|^2 \lesssim s^2\int_q \tilde{\phi}|\psi_x|^2 + s^3\int_q \tilde{\phi}^3|\psi|^2.$$

Then, we have that $\psi_t = e^{-s\tilde{\alpha}}(\tilde{\varphi}_t - s\tilde{\alpha}_t\tilde{\varphi})$, hence

$$\begin{aligned} &s^{-1}\int_q \tilde{\phi}^{-1}e^{-2s\tilde{\alpha}}|\tilde{\varphi}_t|^2 \lesssim s^{-1}\int_q \tilde{\phi}^{-1}|\psi_t|^2 + s\tilde{T}^2\int_q \tilde{\phi}^3|\psi|^2 \\ &\lesssim s^{-1}\int_q \tilde{\phi}^{-1}|\psi_t|^2 + s^3\int_q \tilde{\phi}^3|\psi|^2 \end{aligned}$$

for $s \gtrsim \tilde{T}$. Analogously, one can prove that

$$s^{-1}\int_0^{\tilde{T}}\int_q \tilde{\phi}^{-1}e^{-2s\tilde{\alpha}}|\tilde{\varphi}_{xx}|^2 \lesssim s^{-1}\int_q \tilde{\phi}^{-1}|\psi_t|^2 + s^3\int_q \tilde{\phi}^3|\psi|^2 + s^2\int_q \tilde{\phi}|\psi_x|^2,$$

for $s \gtrsim \tilde{T}^2$.
We combine this with (20) and we obtain the required result

$$\begin{aligned} &s^3\int_q \tilde{\phi}^3e^{-2s\tilde{\alpha}}|\tilde{\varphi}|^2 + s^2\int_q \tilde{\phi}e^{-2s\tilde{\alpha}}|\tilde{\varphi}_x|^2 + s^{-1}\int_q \tilde{\phi}^{-1}e^{-2s\tilde{\alpha}}(|\tilde{\varphi}_{xx}|^2 + |\tilde{\varphi}_t|^2) + s^3\int_{\sigma_1}\tilde{\phi}^3e^{-2s\tilde{\alpha}}|\tilde{\varphi}|^2 \\ &\lesssim s^5\int_{\sigma_0}\tilde{\phi}^5e^{-2s\tilde{\alpha}}|\tilde{\varphi}|^2 + (s+\varepsilon)\int_{\sigma_0}\tilde{\phi}e^{-2s\tilde{\alpha}}|\tilde{\varphi}_t|^2, \end{aligned} \tag{21}$$

for $s \gtrsim \tilde{T}(1+\varepsilon^{-1}\tilde{T}) + \varepsilon^{1/3}\tilde{T}^{2/3}$ and using that

$$(s+\varepsilon+\varepsilon^2 s)s^2\tilde{T}^2\int_{\sigma_0}\tilde{\phi}^5e^{-2s\tilde{\alpha}}|\tilde{\varphi}|^2 \lesssim s^5\int_{\sigma_0}\tilde{\phi}^5e^{-2s\tilde{\alpha}}|\tilde{\varphi}|^2$$

which is true for $s \gtrsim \tilde{T} + \varepsilon^{1/3}\tilde{T}^{2/3}$ (recall that $\varepsilon < 1$). □

With this result, we will now finish the proof of Proposition 10.

Estimate of the boundary term In this paragraph we will estimate the boundary term

$$(s+\varepsilon)\int_{\sigma_0}\tilde{\phi}e^{-2s\tilde{\alpha}}|\tilde{\varphi}_t|^2.$$

First, we observe that this integral can be estimated by

$$s^2 \int_{\sigma_0} \tilde{\phi}^2 e^{-2s\tilde{\alpha}} |\tilde{\varphi}_t|^2,$$

for $s \gtrsim (\tilde{T} + \tilde{T}^2)$. After an integration by parts in time, we get

$$\begin{aligned} s^2 \int_{\sigma_0} \tilde{\phi}^2 e^{-2s\tilde{\alpha}} |\tilde{\varphi}_t|^2 &= \frac{1}{2} s^2 \int_{\sigma_0} (\tilde{\phi}^2 e^{-2s\tilde{\alpha}})_{tt} |\tilde{\varphi}|^2 - s^2 \int_{\sigma_0} \tilde{\phi}^2 e^{-2s\tilde{\alpha}} \tilde{\varphi}\, \tilde{\varphi}_{tt} \\ &\lesssim s^4 \int_{\sigma_0} \tilde{\phi}^6 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|^2 + s^2 \int_{\sigma_0} \tilde{\phi}^2 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|\, |\tilde{\varphi}_{tt}| \end{aligned} \tag{22}$$

In order to estimate the second time derivative at $x = 0$, we will apply some a priori estimates for the adjoint system. Indeed, let us consider the following function

$$\zeta(t, x) := \theta(t) \tilde{\varphi}_t := e^{-s\tilde{\alpha}(t,-L)} \tilde{\phi}^{-5/2}(t, -L) \tilde{\varphi}_t(t, x).$$

Then, this function fulfills the following system:

$$\begin{cases} \zeta_t + \varepsilon^{-1} \zeta_x + \zeta_{xx} = \theta_t \tilde{\varphi}_t & \text{in } (0, \tilde{T}) \times (-L, 0), \\ \varepsilon^2 (\zeta_t - \varepsilon^{-1} \partial_\nu \zeta) - \zeta = \varepsilon^2 \theta_t \tilde{\varphi}_t & \text{on } (0, \tilde{T}) \times \{0\}, \\ \zeta_t - \varepsilon^{-1} \partial_\nu \zeta = \theta_t \tilde{\varphi}_t & \text{on } (0, \tilde{T}) \times \{-L\}, \\ \zeta_{|t=T} = 0 & \text{in } (-L, 0). \end{cases}$$

Using Remark 4 for $(t, x) \longmapsto \zeta(\varepsilon t, x)$ we find in particular

$$\varepsilon \int_\sigma |\zeta_t|^2 \lesssim \int_q |\theta_t|^2 |\tilde{\varphi}_t|^2 + \varepsilon \int_\sigma (\theta_t)^2 |\tilde{\varphi}_t|^2.$$

This directly implies that

$$\varepsilon \int_\sigma \theta^2 |\tilde{\varphi}_{tt}|^2 \lesssim \int_q |\theta_t|^2 |\tilde{\varphi}_t|^2 + \varepsilon \int_\sigma (\theta_t)^2 |\tilde{\varphi}_t|^2.$$

Integrating by parts in time in the last integral, we have

$$\varepsilon \int_\sigma \theta^2 |\tilde{\varphi}_{tt}|^2 \lesssim \left(\int_q |\theta_t|^2 |\tilde{\varphi}_t|^2 + \varepsilon \int_\sigma ((\theta_t)^2)_{tt} |\tilde{\varphi}|^2 + \varepsilon \int_\sigma (\theta_t)^4 \theta^{-2} |\tilde{\varphi}|^2 \right) + \frac{\varepsilon}{2} \int_\sigma \theta^2 |\tilde{\varphi}_{tt}|^2.$$

From the definition of $\theta(t)$ and multiplying the previous inequality by s^{-1}, we find that

$$\varepsilon s^{-1} \int_\sigma (e^{-2s\tilde{\alpha}} \tilde{\phi}^{-5})(., -L) |\tilde{\varphi}_{tt}|^2 \lesssim \tilde{T}^2 \left(s \int_q \tilde{\phi}^{-1} e^{-2s\tilde{\alpha}} |\tilde{\varphi}_t|^2 + \varepsilon (1 + \tilde{T}^2) s^3 \int_\sigma (\tilde{\phi}^3 e^{-2s\tilde{\alpha}})(., -L) |\tilde{\varphi}|^2 \right). \tag{23}$$

Using Cauchy-Schwarz inequality in the last term of the right hand side of (22), we obtain

$$s^2 \int_{\sigma_0} \tilde{\phi}^2 e^{-2s\tilde{\alpha}} |\tilde{\varphi}|\, |\tilde{\varphi}_{tt}| \lesssim \varepsilon \tilde{T}^{-2} s^{-1} \int_\sigma (e^{-2s\tilde{\alpha}} \tilde{\phi}^{-5})(., -L) |\tilde{\varphi}_{tt}|^2 + \varepsilon^{-1} \tilde{T}^2 s^5 \int_{\sigma_0} e^{-4s\tilde{\alpha} + 2s\tilde{\alpha}(., -L)} \tilde{\phi}^9 |\tilde{\varphi}|^2$$

Combining this with (23) and (21) yields the desired inequality (12). $\square$

3 Observability and control

In this section, we prove Theorem 1.

3.1 Dissipation and observability result

Our first goal will be to get some dissipation result.

Proposition 13 *For every $\varepsilon \in (0,1)$, for every time $t_1, t_2 > 0$ such that $t_2 - t_1 > L$ and for every weak solution φ of (S'), the following estimate holds true*

$$\|\varphi(t_1)\|_X \leqslant \exp\left\{-\frac{(t_2-t_1-L)^2}{4\varepsilon(t_2-t_1)}\right\}\|\varphi(t_2)\|_X.$$

Proof We first consider a weight function $\rho(t,x) = \exp(\frac{r}{\varepsilon}x_n)$ with $r \in (0,1)$ some constant which will fixed later. We will first treat the strong solutions case and, using a density argument, we will get the weak solutions case.

We multiply the equation satisfied by φ by $\rho\varphi$ and we integrate on Ω. We get the following identity:

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho|\varphi|^2\right) = -\frac{1}{2}\int_\Omega \rho\partial_n(|\varphi|^2) - \varepsilon\int_\Omega \rho\varphi\Delta\varphi.$$

We then integrate by parts in space, which due to $\nabla\rho = \frac{r}{\varepsilon}\rho e_n$, provides

$$-\frac{1}{2}\int_\Omega \rho\partial_n(|\varphi|^2) = \frac{r}{2\varepsilon}\int_\Omega \rho|\varphi|^2 - \frac{1}{2}\left(\int_{\Gamma_0}\rho|\varphi|^2 - \int_{\Gamma_1}\rho|\varphi|^2\right),$$

and

$$-\varepsilon\int_\Omega \rho\varphi\Delta\varphi = \varepsilon\int_\Omega \rho|\nabla\varphi|^2 + \frac{r}{2}\int_\Omega \rho\partial_n(|\varphi|^2) - \varepsilon\left(\int_{\Gamma_0}\rho\varphi\partial_n\varphi - \int_{\Gamma_1}\rho\varphi\partial_n\varphi\right)$$

$$= \varepsilon\int_\Omega \rho|\nabla\varphi|^2 - \frac{r^2}{2\varepsilon}\int_\Omega \rho|\varphi|^2 + \frac{r}{2}\left(\int_{\Gamma_0}\rho|\varphi|^2 - \int_{\Gamma_1}\rho|\varphi|^2\right) - \varepsilon\left(\int_{\Gamma_0}\rho\varphi\partial_n\varphi - \int_{\Gamma_1}\rho\varphi\partial_n\varphi\right).$$

Using now the boundary conditions for φ and summing up these identities, we finally get

$$\frac{d}{dt}\left(\int_\Omega \rho|\varphi|^2\right) \geq \frac{r(1-r)}{\varepsilon}\int_\Omega \rho|\varphi|^2 + (1-r)\int_\Gamma \rho|\varphi|^2 - 2\varepsilon\int_\Gamma \rho\varphi_t\varphi.$$

On the other hand, it is straightforward that

$$\frac{d}{dt}\left(\varepsilon\int_\Gamma \rho|\varphi|^2\right) = 2\varepsilon\int_\Gamma \rho\varphi_t\varphi,$$

and, consequently, using that $r \in (0,1)$, we have obtained

$$\frac{d}{dt}\left(\|\sqrt{\rho(.)}\varphi(t)\|_X^2\right) \geq \frac{r(1-r)}{\varepsilon}\|\sqrt{\rho(.)}\varphi(t)\|_X^2.$$

Gronwall's lemma combined with $\exp(-\frac{r}{\varepsilon}L) \leqslant \rho(\cdot) \leqslant 1$ successively gives

$$\|\sqrt{\rho(\cdot)}\varphi(\cdot)\|_X^2 \leqslant \exp\left(-\frac{r(1-r)}{\varepsilon}(t_2-t_1)\right)\|\sqrt{\rho(\cdot)}\varphi(t_1)\|_X^2$$

and

$$\|\varphi(t_1)\|_X^2 \leqslant \exp\left(-\frac{1}{\varepsilon}\left(r(1-r)(t_2-t_1) - rL\right)\right)\|\varphi(t_2)\|_X^2$$

We finally choose

$$r := \frac{t_2-t_1-L}{2(t_2-t_1)} \in (0,1),$$

which gives the result. □

We will now use this dissipation estimate with our Carleman inequality to get the desired result.

Proposition 14 *If $n = 1$ and for $\frac{T}{L} - 1$ sufficiently large and ε sufficiently small, the observability constant $C_T(\varepsilon)$ is bounded than*

$$C \exp\left(-\frac{k}{\varepsilon}\right)$$

where C, k are some positive constants.

Proof

- We begin by estimating both sides of the Carleman inequality obtained above. We use the same notations as above and we define $m = C - e^{2L}$ and $M = C - e^{L}$. We first get

$$s^9 \int_0^T e^{-4s\alpha(.,0)+2s\alpha(.,-L)}(\phi^9|\varphi|^2)(.,0) \lesssim s^9(\varepsilon T)^{-18} \exp\left(\frac{s(2M-4m)}{(\varepsilon T)^2}\right) \int_0^T |\varphi|^2(.,0).$$

  On the other hand, using that $\phi \gtrsim \dfrac{1}{(\varepsilon T)^2}$ on $[\frac{T}{4}, \frac{3T}{4}]$, we have the following estimate from below for the left hand-side of the Carleman inequality (12)

$$\frac{s^3}{(\varepsilon T)^6} \exp\left(-\frac{2sM}{(\varepsilon T)^2}\right)\left(\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{-L}^{0} |\varphi|^2 + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\{-L,0\}} |\varphi|^2\right).$$

  Consequently we get that

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \|\varphi(t)\|_X^2 dt \lesssim s^6(\varepsilon T)^{-12} \exp\left(\frac{4s(M-m)}{(\varepsilon T)^2}\right) \int_0^T |\varphi|^2(.,0) := C \int_0^T |\varphi|^2(.,0).$$

  We now choose $s \backsim (\varepsilon T)^2 + (\varepsilon T)$. The above constant C is consequently estimated by

$$\varepsilon^{-11} e^{c/\varepsilon} \lesssim e^{c'/\varepsilon}$$

  for $c' > c$ and c well-chosen.

- We now deduce the result using dissipation estimates. We have just proven

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \|\varphi(t)\|_X^2 dt \lesssim e^{\frac{c'}{\varepsilon}} \int_0^T \int_{\Gamma_0} |\varphi|^2.$$

  We now use the dissipation property with $t_1 = 0$ and $t_2 = t \in \left]\frac{T}{4}, \frac{3T}{4}\right[$. We easily get, provided $T > 4L$,

$$\frac{T}{2} \exp\left(\frac{(T-4L)^2}{8\varepsilon T}\right) \|\varphi(0)\|_X^2 \leqslant \int_{\frac{T}{4}}^{\frac{3T}{4}} \|\varphi(t)\|_X^2 dt,$$

  which gives the result with $k = \dfrac{1}{8}\left(1 - \dfrac{4L}{T}\right)(T-4L) - c' > 0$ provided that $\dfrac{T}{L} > 8 + 32c'$.

$\square$

3.2 Proof of Theorem 1

To show the controllability result for (S_v), we will adopt some minimization strategy inspired by the classical heat equation.

Proposition 15 *A necessary and sufficient condition for the solution of problem (S_v) to satisfy $u(T) = 0$ is given by:*

$$\forall \varphi_T \in X, < \varphi(0), u_0 >_X = \int_0^T \int_{\Gamma_0} \varphi v.$$

where φ is the solution of problem (S') with final value φ_T.

Proof We apply the definition of solution to u against φ strong solutions of (S'), which gives

$$\int_0^T \int_{\Gamma_0} \varphi v = \varepsilon \int_{\partial\Omega} \varphi(0) u_0 - \varepsilon \int_{\partial\Omega} \varphi_T u(T) + \int_{\Omega} \varphi(0) u_0 - \int_{\Omega} \varphi_T u(T)$$

and we get the desired equivalence by approximation of weak solutions by strong solutions. □

Theorem 16 *The following properties are equivalent*

- $\exists C_1 > 0, \forall \varphi_T \in X$; $\|\varphi(0)\|_X \leq C_1 \|\varphi\|_{L^2((0,T)\times\Gamma_0)}$ *where* φ *is the solution of problem* (S'),
- $\exists C_2 > 0, \forall u_0 \in X, \exists v \in L^2((0,T)\times\Gamma_0)$ *such that* $\|v\|_{L^2((0,T)\times\Gamma_0)} \leqslant C_2 \|u_0\|_X$ *and the solution* u *of problem* (S_v)*satisfies* $u(T) = 0$.

Moreover, $C_1 = C_2$.

Proof $(\Rightarrow)$ Let $u_0 \in X$. We define H as the closure of X for the norm defined by

$$\|\varphi_T\|_H = \left(\int_0^T \int_{\Gamma_0} |\varphi(t,x)|^2 d\sigma dt \right)^{\frac{1}{2}},$$

where φ is the corresponding solution of (S'). Using the observability assumption and backward uniqueness (Proposition 8), one sees that it is indeed a norm on X.

We define a functional J in the following way

$$J(\varphi_T) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \varphi^2(t,x) d\sigma dt - \int_{\Omega} u_0(x) \varphi(0,x) dx - \varepsilon \int_{\partial\Omega} u_0(x) \varphi(0,x) d\sigma.$$

J is clearly convex and our assumption imply that J is continuous on H. Moreover, thanks to our observability assumption, J is coercive. Indeed, one has:

$$J(\varphi_T) \geqslant \frac{1}{2} \|\varphi_T\|_H^2 - C \|\varphi_T\|_H$$

for $\varphi_T \in H$.
Thus J possesses a global minimum $\widehat{\varphi}_T \in H$, which gives, writing Euler-Lagrange equations,

$$\forall \varphi_T \in H, \int_0^T \int_{\Gamma_0} \varphi \hat{\varphi} = \int_{\Omega} u_0 \varphi(0) + \varepsilon \int_{\partial\Omega} u_0 \varphi(0). \tag{24}$$

According to Proposition 15, we have shown the existence of an admissible control defined by $v = \hat{\varphi}_{|(0,T)\times\Gamma_0}$. Moreover, choosing $\varphi_T = \widehat{\varphi}_T$ in (24), we obtain the following estimate

$$\|v\|^2_{L^2((0,T)\times\Gamma_0)} \leqslant \left(\int_{\Omega} u_0^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} \hat{\varphi}^2(0) \right)^{\frac{1}{2}} + \varepsilon \left(\int_{\partial\Omega} u_0^2 \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \hat{\varphi}^2(0) \right)^{\frac{1}{2}}$$

Using our hypothesis, we are done.

$(\Leftarrow)$ If v is an admissible control with continuous dependence on u_0, Proposition 15 gives us, for every $\varphi_T \in X$,

$$\int_{\Omega} u_0 \varphi(0) + \varepsilon \int_{\partial\Omega} u_0 \varphi(0) = \int_0^T \int_{\Gamma_0} \varphi v.$$

Choosing now $u_0 = \varphi(0)$ gives us the estimate

$$\|\varphi(0)\|_X^2 \leqslant C_2 \|\varphi_T\|_H \|u_0\|_X$$

that is

$$\|\varphi(0)\|_X \leqslant C_2 \|\varphi_T\|_H.$$

□

Conclusion and open problems

The question of controllability in higher dimension is open and seems to be hard to obtain using Carleman estimates. Another approach to obtain the observability estimate in the n-dimensional case could be similar to the one of Miller (see [9]) but it seems that our problem is not adapted to that framework (one can check that the link between the one-dimensional problem and the general one is not as simple as it may seem).

If $n = 1$, one can wonder what is the minimal time to get a vanishing control cost when the viscosity goes to zero. The intuitive result would be L, but it seems that Carleman estimates can not give such a result.

References

[1] Coron, J.-M., 2007, Control and nonlinearity. *Mathematical Surveys and Monographs, 136, AMS.*

[2] Coron, J.-M., Guerrero, S., 2005, Singular optimal control: a linear 1-D parabolic-hyperbolic example. *Asymptot. Anal.*, **44(3-4)**, 237-257.

[3] Dolecki, S., Russel, D., 1977, A general theory of observation and control. *SIAM J. Control and Optimization*, **15**, No 2, 185-220.

[4] Fursikov, A. V., Imanuvilov, O., 1996, Controllability of evolution equations. **Lecture Notes Series, 34**. *Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul.*

[5] Glass, O.,Guerrero,S., 2007, On the uniform controllability of the Burgers equation. *SIAM J. Control and Optimization*, **46**, No 4, 1211-1238.

[6] Halpern, L., 1986, Artificial boundary for the linear advection diffusion equation. *Mathematics of Computation*, **46**, No 174, 425-438.

[7] Lions, J.-L., Magenes, E., 1968, Problèmes aux limites non homogènes et applications, *Vol. I, II.* (French) *Dunod, Paris.*

[8] Lions, J.-L., Malgrange, B., 1960, Sur l'unicité rétrograde dans les problèmes mixtes paraboliques. (French) *Math. Scand.*, **8**, 277–286.

[9] Miller, L., 2005, On the null-controllability of the heat equation in unbounded domains. *Bulletin des Sciences Mathématiques*, **129**, No 2, 175-185.

[10] Temam, R.,, 1977, Navier-Stokes equations. Theory and numerical analysis. *Studies in Mathematics and its Applications, Vol. 2, North-Holland.*